VIII. "Note of a Theory of Orthoptic and Isoptic Loci." By CHARLES TAYLOR, D.D., Master of St. John's College, Cambridge. Communicated by J. W. L. GLAISHER, M.A., F.R.S. Received June 10, 1884.

The orthoptic locus of a curve and its isoptic loci are the loci of the points of concourse of pairs of tangents drawn to it at right angles, and at angles equal to given angles, respectively.

As a step towards a general theory of such loci, of which special cases only have been treated hitherto, it is shown below that the order of the orthoptic locus of a curve of class n is $n(n-1)$, and the order of its isoptic loci $2n(n-1)$.

The principle on which our proof depends is that lines drawn from either of the circular points at infinity I and J may be regarded as intersecting at any angle whatsoever,* but such as are drawn from any other point at infinity, real or imaginary, can only be regarded as parallel, unless one of them be the straight line at infinity, which makes an indeterminate angle with any straight line.

1. *The Orthoptic Locus of a Curve of any Class.*

To a curve of the nth class n tangents, constituting $\frac{1}{2}n(n-1)$ quasi-orthogonal pairs, can be drawn from I or J. Each of these is therefore a point of the order $\frac{1}{2}n(n-1)$ on the orthoptic locus, and this locus, having in general no other points at infinity, is of the order $n(n-1)$.

If the curve touches the line IJ in one point, $n-1$ other tangents can in general be drawn to it from any point on IJ, and each of them may be regarded as orthogonal to IJ. Every point at infinity is therefore of the order $n-1$ on the orthoptic locus, and the remainder of the locus when the factor IJ^{n-1} is subtracted is of the order $n(n-1)-(n-1)$, that is to say $(n-1)^2$, and contains I and J as points of the order $\frac{1}{2}n(n-1)-(n-1)$, or $\frac{1}{2}(n-1)(n-2)$.

If the curve touches IJ in r points it appears in like manner that the orthoptic locus contains IJ as a factor $r(n-1)$ times, and the remainder of the locus is therefore of the order $(n-r)(n-1)$, and contains I and J as points of the order $\frac{1}{2}n(n-1)-r(n-1)$, or $\frac{1}{2}(n-2r)(n-1)$.

* To demonstrate the existence of the circular points, draw a circle, and upon it take an arc AB at random, and let x be either of the points in which the circle meets the line at infinity. Any two straight lines through x may be regarded as making zero angles with xA and xB respectively, and therefore as including an angle equal to that standing on the arc AB, which may be of any magnitude whatsoever. It readily follows that all circles pass through x, and hence that there can be only two such points on the line at infinity.

Notice in verification the case of the conics (de la Hire, 1685), and likewise that of the cardioid, whose orthoptic locus consists of a circle and a bicircular quartic, which together make up a tricircular sextic. When the curve resolves itself into n point-factors the orthoptic locus evidently consists of the $\frac{1}{2}n(n-1)$ circles described on the lines joining the points two and two as diameters.

2. *Pedals of a Pair of Curves.*

The locus of the vertex of a right angle whose arms envelope two curves of class m and class n respectively may be called the pedal of the two curves, or of the one with respect to the other, and the corresponding locus generated by the vertex of any other constant angle may be called a skew pedal of the two curves, or of the one with respect to the other. The former locus becomes a pedal commonly so called when one of the curves degenerates into a point.

From the reasoning used above it is evident that the pedal of two such curves is an mn-circular $2mn$-ic.

This may also be deduced from the formula for the orthoptic locus as follows: The two curves make up a curve of class $m+n$, whose orthoptic locus is the aggregate of the pedal and the orthoptic loci of the two curves. The pedal is therefore of the order

$$(m+n)(m+n-1)-m(m-1)-n(n-1),$$

that is to say, it is of the order $2mn$, and it contains I and J as points of the order mn.

3. *Isoptic Loci and their Reciprocals.*

a. Any two tangents to a curve from I or J may be regarded as intersecting at angles α and $\pi-\alpha$ or these reversed, and their point of concourse thus belongs doubly to the corresponding isoptic locus. The order of such loci is therefore double of that of the orthoptic locus, and they pass twice as often through I and J.

For example—

(1.) The ellipse, to which one pair of tangents only can be drawn from I or J, may be regarded as subtending any angle or its supplement at those points. These are therefore double points on the corresponding isoptic locus, which is accordingly a bicircular quartic.

(2.) The parabola may be regarded as subtending any angle or its supplement at every point on the line at infinity. Its isoptic loci therefore contain the factor IJ^2, and the remainders, when this factor is rejected, are hyperbolas (or ellipses).

(3.) It may be deduced from the formula for isoptic loci, or proved directly by the method used above, that the skew pedals of a pair of curves of class m and class n respectively are $2mn$-circular,

$4mn$-ics. Thus the skew pedals of an ellipse with respect to a point (regarded as a curve of the first class) are of the eighth order, each consisting, of course, of two equal curves similar to the right pedal.

By taking a pair of lines drawn at random through either circular point, which may be regarded as inclined at an indeterminate angle, and supposing them to coalesce, we infer that any straight line through I or J may be regarded as making an indeterminate angle with itself.*

Hence the points of contact of the tangents from I and J to any curve are points on its orthoptic locus, and they are doubly points on its isoptic loci.

In the case of the conics these are the only points in which such loci meet the curve. If, therefore, $U \equiv \phi(x, y) = 0$ be a conic, and $u = 0$ its orthoptic locus, the bicircular quartics which are its isoptic loci will be represented by

$$U - k \,.\, u^2 = 0,$$

where k is a constant which vanishes when the isoptic angle is zero, in which case the locus consists of the conic and the line at infinity, and is infinite when the angle is a right angle, the isoptic being then the orthoptic locus.

Reciprocally, in a curve of the nth order, if a chord subtends a constant angle at a fixed point its envelope is of the class $2n(n-1)$.

The various points in the theory of plane orthoptic and isoptic loci propounded in this note have been verified by analytical methods in an unpublished paper by Mr. J. S. Yeo, Fellow of St. John's College.

The following notes on isoptic and other loci in space are taken from a valuable and suggestive series of investigations by Mr. Joseph Larmor, Fellow of St. John's College.

A solid has in general *six* degrees of freedom to move. The corner of a cube whose three faces are constrained to touch a surface loses three and retains *three*, and the locus of a point rigidly connected with it is not a surface, but a solid bounded by a certain envelope. When the cube-angle envelopes a quadric it can enjoy one of its degrees of freedom without displacement of its vertex, for if a cone of the second degree has one triad of orthogonal tangent planes, it has a

* It is sometimes said that such lines are at right angles to themselves; but this statement, although true so far as it goes, is inadequate. The angle between lines parallel to $y + mx = 0$ and $y + m'x = 0$ is $\tan^{-1}\dfrac{m \backsim m'}{1 + mm'}$, and when $m = m' = \sqrt{-1}$, it is $\tan^{-1}\dfrac{0}{0}$, the numerator as well as the denominator vanishing.

singly infinite number: consequently the locus of the vertex contracts in this case into a surface, the orthoptic sphere. The locus of the vertex of a trihedral angle which envelopes a quadric is in general the space bounded by two surfaces.

Next consider a complex of lines of the nth order; those of its lines which pass through a specified point form a cone of the nth order; this cone can be circumscribed by a cube-angle provided the point lies within a certain solid space. When the complex is of the second order the solid locus degenerates into a surface, which is a quartic passing through the imaginary circle at infinity; and when the complex is composed of the tangent lines of a surface of the second order the locus is made up of the surface and its orthoptic sphere.

Similar considerations hold for the locus of the point of concourse of a triad of tangent lines at right angles. When the complex is of the second order the locus degenerates into a quadric.

Mr. Larmor has briefly considered the problem of a surface constrained to touch three surfaces, deducing as a special case that the locus of the vertex of a cube-angle whose faces touch a quadric or three confocal quadrics is a sphere.

IX. "On a New Form of Voltaic Battery." By PAUL JABLOCHKOFF. Communicated by WARREN DE LA RUE, M.A., D.C.L., Ph.D., F.R.S. Received May 12, 1884.

The battery which I have the honour to bring under the notice of the Royal Society is one of high electromotive force, namely, about two and three quarter volts, and a single cell consequently decomposes water; it is very light and portable, and convenient for many purposes. The electro-positive element is sodium, the electro-negative element is either carbon, spongy platinum, copper, or other metallic gauze; no fluid is used in which to immerse the plates, but the atmospheric air which is always impregnated with more or less hygrometric moisture serves to set up the action of the battery by giving up sufficient moisture to wet the surface of the sodium, so that a very thin film of fluid (a solution of soda), is thus interposed between the sodium and electro-negative element, and the internal resistance is very small in consequence of the thinness of the film of fluid. The sodium is used in the form of plates, conveniently about a quarter of an inch thick, and the plates of carbon, of which one is placed on each side of the sodium, a little longer and about the same thickness as the sodium; these plates, carbon, sodium, carbon, are kept together by means of vulcanised rubber bands, and suspended vertically, a vessel being placed underneath to receive the soda solution as it forms.